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Statistical moments of the random linear transport equation

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ABSTRACT

This paper deals with a numerical scheme to approximate the mth moment of the solution of the one-dimensional random linear transport equation. The initial condition is assumed to be a random function and the transport velocity is a random variable. The scheme is based on local Riemann problem solutions and Godunov's method. We show that the scheme is stable and consistent with an advective–diffusive equation. Numerical examples are added to illustrate our approach.

1. Introduction

Partial differential equations have been important models during the last centuries, mainly because they have the fundamental support of differential calculus, numerical methods, and computers. However, the formulation of a physical process as a partial differential equation demands experiments to measure the data, for example, the diffusion coefficient, permeability of a porous media, initial conditions, boundary conditions and so on. This means that the interpretation of the data as random variables is more realistic in some practical situations. Differential equations with random parameters are called Random Differential Equations; new mathematical methods have been developed to deal with this kind of problems (see [6,9,13,16], for example).

We are interested in the solution of the random linear transport equation

$$\begin{cases} Q_t(x,t) + AQ_x(x,t) = 0, & t > 0, \quad x \in \mathbb{R}, \\ Q(x,0) = Q_0(x), \end{cases} \tag{1}$$

where A is a random variable and $Q_0(x)$ is a random function.

According to [1], the solution for the random Riemann problem (1) with

$$Q_0(x) = \begin{cases} Q_L & \text{if} \quad x < 0, \\ Q_R & \text{if} \quad x > 0, \end{cases} \tag{2}$$

where Q_L and Q_R are random variables, is given by

$$Q(x,t) = Q_L + X\left(\frac{x}{t}\right)(Q_R - Q_L). \tag{3}$$

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In (3) X is the Bernoulli random variable with $P\{X(\xi)=1\}=F_A(\xi)$ where F_A is the cumulative probability function of A. Furthermore, in case of independence between A and both Q_L and Q_R, the mth moment of $Q(x,t)$, $\langle Q^m(x,t)\rangle$, $m\in\mathbb{N}$, $m\geqslant 1$, is given by

$$\langle Q^m(x,t)\rangle = \langle Q_L^m\rangle + F_A\left(\frac{x}{t}\right)\left[\langle Q_R^m\rangle - \langle Q_L^m\rangle\right]. \tag{4}$$

The closed solution (3) and Godunov's ideas [7,10,11] are used in [2] and [4] to design numerical methods to compute the mean and the variance of the solution to (1). These methods are explicit and neither demand generation of random numbers (as does the Monte Carlo method [5,12,15,17]), nor require differential equations governing the statistical moments (as in the effective equations methodology [6,17]). Moreover, the schemes are stable and consistent with an advective–diffusive equation which agrees with the effective equation to the expectation presented in the literature (see [6], for example). In [3] we use the idea of collecting deterministic realizations through their probability functions to solve the nonlinear random Riemann–Burgers equation.

In this paper, we deal with the general moments of the solution to (1). The outline of this paper is as follows. In Section 2 we use (3) and (4) to design a numerical method to the mth statistical moment of the solution to the general problem (1). We present the CFL condition under which the local solutions do not interact between themselves. In Section 3 we show the stability of the numerical scheme and its consistency with an advective–diffusive equation. We show that the diffusion coefficient is related with the probability density function of the velocity by Eq. (18), which has a simple solution in the normal velocity case. Furthermore, in Section 4 we present a decoupled system of partial differential equations to be satisfied by the central moments of the random solution. All the partial differential equations in this paper are linear. In fact, denoting by $\mathbb{L}(u)=u_t+\langle A\rangle u_x-\nu u_{xx}$, the equations are: $\mathbb{L}(u)=0$, for the moments, and $\mathbb{L}(u)=f$, for the central moments. Computational experiments and comparisons with the Monte Carlo method are presented in Section 5.

2. The numerical scheme

In this section, we present the numerical method for the mth statistical moment of the solution to (1). The method is based on the juxtaposition of Riemann problems whose solutions are given by (3). We discretize both space and time assuming a uniform mesh spacing: $x_j=j\Delta x$, $x_{j\pm1/2}=x_j\pm(\Delta x/2)$, $t_n=n\Delta t$, $t_{n\pm1/2}=t_n\pm(\Delta t/2)$, for $\Delta x,\Delta t>0$. In Fig. 1 we present a schematic diagram of the algorithm. Let us assume that the random variables Q_j^n and the mth moments $\left\langle Q_j^{m,n}\right\rangle=\langle Q^m(x_j,t_n)\rangle$ are known at $t=t_n$.

In the following we use the ideas of Reconstruct-Evolve-Average (REA), algorithm [7,11] to approximate $\left\langle Q_j^{m,n+1}\right\rangle=\langle Q^m(x_j,t_{n+1})\rangle$.

Step 1 We reconstruct the piecewise random constant function $\widetilde{Q}(x,t_n)$ from Q_j^n, i.e, $\widetilde{Q}(x,t_n)=Q_j^n$ for $x\in[x_{j-1/2},x_{j+1/2}]$. The piecewise constant random function $\widetilde{Q}(x,t_n)$ defines a set of local random Riemann problems, each one centered at $x=x_{j-1/2}$,

$$\begin{aligned}&Q_t(x,t)+AQ_x(x,t)=0,\quad t>t_n,\quad x\in\mathbb{R},\\&Q(x,t_n)=\begin{cases}Q_{j-1}^n, & \text{if}\quad x<x_{j-1/2},\\ Q_j^n, & \text{if}\quad x>x_{j-1/2}.\end{cases}\end{aligned} \tag{5}$$

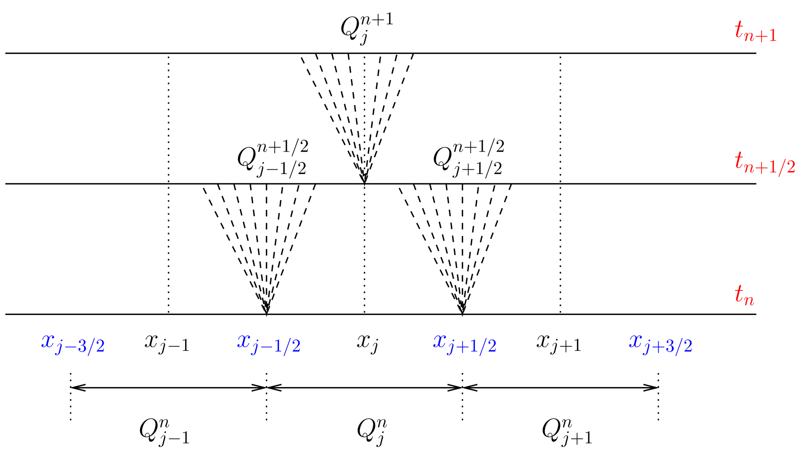

Fig. 1. Schematic diagram of the algorithm.

Step 2 From (3) and (4), the local solutions of (5) and the respective statistical moments are given by

$$G_{j-1/2}(x,t_{n+1/2}) = Q^n_{j-1} + X\left(\frac{x-x_{j-1/2}}{\Delta t/2}\right)\left[Q^n_j - Q^n_{j-1}\right] \tag{6}$$

and

$$\left\langle G^m_{j-1/2}(x,t_{n+1/2})\right\rangle = \left\langle Q^{m,n}_{j-1}\right\rangle + F_A\left(\frac{x-x_{j-1/2}}{\Delta t/2}\right)\left[\left\langle Q^{m,n}_j\right\rangle - \left\langle Q^{m,n}_{j-1}\right\rangle\right]. \tag{7}$$

The global solution at $t = t_{n+1/2}$, $\widetilde{Q}(x,t_{n+1/2})$, can be constructed by piecing together the local random Riemann solutions (6), provided that $\Delta t/2$ is sufficiently small such that adjacent local random Riemann solutions do not interact. Therefore, taking into account the similarity property of the random Riemann solutions, Δx and Δt must be chosen such that

$$G_{j-1/2}(x,t_{n+1/2})|_{x=x_{j-1}} \approx Q^n_{j-1} \quad \text{and} \quad G_{j-1/2}(x,t_{n+1/2})|_{x=x_j} \approx Q^n_j,$$

where the symbol "≈" means "*sufficiently near to*". By substituting these conditions in (6) we must have

$$F_A\left(-\frac{\Delta x}{\Delta t}\right) \approx 0 \quad \text{and} \quad F_A\left(\frac{\Delta x}{\Delta t}\right) \approx 1. \tag{8}$$

Remark 1. We may regard (8) as the CFL condition for the method: the interval $[-\Delta x/\Delta t, \Delta x/\Delta t]$ must contain an effective support of the density probability function of A. This means that the probability of A outside of the interval $[-\Delta x/\Delta t, \Delta x/\Delta t]$ is sufficiently near to zero, and then may be disregarded. The existence of an effective support is ensured by Chebyshev's inequality: $P\{|A - \langle A\rangle| \geqslant k\sigma_A\} \leqslant 1/k^2$, for all $k > 0$, where σ_A is the standard variation of A. If we take $1/k^2$ sufficiently close to zero, to escape from the interaction between solutions of Riemann problems we must take $(|\langle A\rangle| + k\sigma_A)\Delta t/\Delta x \leqslant 1$.

Under condition (8) we conclude Step 2 by taking

$$\widetilde{Q}(x,t_{n+1/2}) = \sum_{j-1/2} G_{j-1/2}(x,t_{n+1/2})\ \mathbf{1}_{[x_{j-1},x_j]}$$

where $\mathbf{1}_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$. From (7) it follows that

$$\left\langle \widetilde{Q}^m(x,t_{n+1/2})\right\rangle = \sum_{j-1/2}\left\langle G^m_{j-1/2}(x,t_{n+1/2})\right\rangle\ \mathbf{1}_{[x_{j-1},x_j]}. \tag{9}$$

In a similar way, using the values at $t = t_{n+1/2}$, we obtain

$$\left\langle \widehat{Q}^m(x,t_{n+1})\right\rangle = \sum_j \left\langle G^m_j(x,t_{n+1})\right\rangle\ \mathbf{1}_{[x_{j-1/2},x_{j+1/2}]}. \tag{10}$$

Step 3 We use (10) to approximate $\left\langle Q^{m,n+1}_j\right\rangle$ as the average value of $\left\langle \widehat{Q}^m(x,t_{n+1})\right\rangle$ over the interval $[x_{j-1/2}, x_{j+1/2}]$:

$$\begin{aligned}\left\langle Q^{m,n+1}_j\right\rangle &\simeq \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}}\left\langle \widehat{Q}^m(x,t_{n+1})\right\rangle dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}}\left\langle G^m_j(x,t_{n+1})\right\rangle dx\\ &= \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}}\left\{\left\langle Q^{m,n+1/2}_{j-1/2}\right\rangle + F_A\left(\frac{x-x_j}{\Delta t/2}\right)\left[\left\langle Q^{m,n+1/2}_{j+1/2}\right\rangle - \left\langle Q^{m,n+1/2}_{j-1/2}\right\rangle\right]\right\}dx\\ &= \left\langle Q^{m,n+1/2}_{j-1/2}\right\rangle + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx\right\}\left[\left\langle Q^{m,n+1/2}_{j+1/2}\right\rangle - \left\langle Q^{m,n+1/2}_{j-1/2}\right\rangle\right].\end{aligned} \tag{11}$$

Likewise, we use (9) to approximate $\left\langle Q^{m,n+1/2}_{j-1/2}\right\rangle$:

$$\begin{aligned}\left\langle Q^{m,n+1/2}_{j-1/2}\right\rangle &\simeq \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j}\left\langle \widetilde{Q}^m(x,t_{n+1/2})\right\rangle dx = \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j}\left\langle G^m_{j-1/2}(x,t_{n+1/2})\right\rangle dx\\ &= \frac{1}{\Delta x}\int_{x_{j-1}}^{x_j}\left\{\left\langle Q^{m,n}_{j-1}\right\rangle + F_A\left(\frac{x-x_{j-1/2}}{\Delta t/2}\right)\left[\left\langle Q^{m,n}_j\right\rangle - \left\langle Q^{m,n}_{j-1}\right\rangle\right]\right\}dx\\ &= \left\langle Q^{m,n}_{j-1}\right\rangle + \frac{\Delta t}{2\Delta x}\left\{\int_{-\frac{\Delta x}{\Delta t}}^{\frac{\Delta x}{\Delta t}} F_A(x)dx\right\}\left[\left\langle Q^{m,n}_j\right\rangle - \left\langle Q^{m,n}_{j-1}\right\rangle\right].\end{aligned} \tag{12}$$

The following result is proved in [4]:

Lemma 2. *Let Y be a random variable and $[-\xi,\xi]$ an effective support of the density probability function of Y, i.e., $F_Y(-\xi) \approx 0$ and $F_Y(\xi) \approx 1$. Then*

$$\int_{-\xi}^{\xi} F_Y(x)dx \approx \xi - \langle Y\rangle. \tag{13}$$

Inserting (13) in (11) and (12), and denoting $\lambda = \Delta t\langle A\rangle/\Delta x$, gives

$$\left\langle Q_j^{m,n+1}\right\rangle = \frac{1}{2}\left[\left\langle Q_{j-1/2}^{m,n+1/2}\right\rangle + \left\langle Q_{j+1/2}^{m,n+1/2}\right\rangle\right] - \frac{\lambda}{2}\left[\left\langle Q_{j+1/2}^{m,n+1/2}\right\rangle - \left\langle Q_{j-1/2}^{m,n+1/2}\right\rangle\right] \tag{14}$$

and

$$\left\langle Q_{j-1/2}^{m,n+1/2}\right\rangle = \frac{1}{2}\left[\left\langle Q_{j-1}^{m,n}\right\rangle + \left\langle Q_j^{m,n}\right\rangle\right] - \frac{\lambda}{2}\left[\left\langle Q_j^{m,n}\right\rangle - \left\langle Q_{j-1}^{m,n}\right\rangle\right]. \tag{15}$$

Grouping these expressions we summarize the two-step scheme (14) and (15) in the one-step explicit method:

$$\left\langle Q_j^{m,n+1}\right\rangle = \left\langle Q_j^{m,n}\right\rangle - \frac{\lambda}{2}\left[\left\langle Q_{j+1}^{m,n}\right\rangle - \left\langle Q_{j-1}^{m,n}\right\rangle\right] + \frac{1}{4}(1+\lambda^2)\left[\left\langle Q_{j+1}^{m,n}\right\rangle - 2\left\langle Q_j^{m,n}\right\rangle + \left\langle Q_{j-1}^{m,n}\right\rangle\right]. \tag{16}$$

Remark 3. The numerical scheme (16) is conservative, i.e., it can be rewritten as

$$\left\langle Q_j^{m,n+1}\right\rangle = \left\langle Q_j^{m,n}\right\rangle - \frac{\Delta t}{\Delta x}\left[F_{j+1/2}^{m,n} - F_{j-1/2}^{m,n}\right],$$

where $F_{j-1/2}^{m,n} = (1/2)\langle A\rangle\left[\left\langle Q_{j-1}^{m,n}\right\rangle + \left\langle Q_j^{m,n}\right\rangle\right] - (1/4)\langle A\rangle(1/\lambda+\lambda)\left[\left\langle Q_j^{m,n}\right\rangle - \left\langle Q_{j-1}^{m,n}\right\rangle\right]$ is an approximation to the average flux at $x = x_{j-1/2}$.

3. Numerical analysis of the scheme

The scheme (16) is a generalization of a previously studied scheme to the mean ($m = 1$) of the solution to (1). Therefore, we can use the same arguments used in [4] to show

- *Consistency*: if $v = \Delta x^2/(4\Delta t)$ is fixed then the numerical scheme (16) yields an $\mathcal{O}(\Delta x^2)$ approximation for the solution of the partial differential equation

$$\begin{aligned} &u_t + \langle A\rangle u_x = v\ u_{xx}, \\ &u(x,0) = \langle Q_0(x)^m\rangle; \end{aligned} \tag{17}$$

- *Stability*: the numerical method (16) is stable under the CFL condition (8).

As a linear problem, the convergence of (16) to the differential equation (17) is a consequence of the Lax Equivalence Theorem, no matter what $v = \Delta x^2/(4\Delta t)$ is. The following proposition gives additional information about the diffusion associated with the random velocity, A.

Proposition 4. *The diffusion coefficient in (17) must satisfy*

$$-f_A'\left(\frac{x}{t}\right)v(x,t) = f_A\left(\frac{x}{t}\right)(x - \langle A\rangle t), \tag{18}$$

where $f_A(\xi) = d[F_A(\xi)]/d\xi$ is the density probability function of A.

Proof. As a general differential equation, (17) must be satisfied by every particular solution. The random Riemann problem (1)–(2) is a particular case of (1) with known moments given by (4):

$$\langle Q^m(x,t)\rangle = \langle Q_L^m\rangle + F_A\left(\frac{x}{t}\right)[\langle Q_R^m\rangle - \langle Q_L^m\rangle].$$

Direct derivations and substitution of this solution in (17) gives (18), a necessary condition to $v(x,t)$. □

3.1. The normal case

Let $A \sim N(\langle A\rangle, \sigma_A)$. Using the normal probability density function in (18) we obtain $v = \sigma_A^2 t$. In this case, the differential equation (17) turns to be

$$\begin{aligned} &u_t + \langle A\rangle u_x = (\sigma_A^2 t)u_{xx}, \quad t > 0, \\ &u(x,0) = \langle Q_0(x)^m\rangle, \end{aligned} \tag{19}$$

which agrees with the effective equation for the statistical mean presented by some authors (see [6], for example). We emphasize that our convergence results show that the differential equation which describes the evolution of all the moments is the same. Using (18) we may also show that if $v(x,t)$ depends only on t then A is normally distributed.

Now we use the consistency condition to define proper mesh spacing. Let $t = t_f$ be fixed, and select Δt and Δx such that

$$\frac{\Delta x^2}{4\Delta t} = \nu = \frac{1}{2}(\sigma_A^2 t_f). \tag{20}$$

The convergence results show that our method converges to the solution of the differential equation

$$\begin{aligned} &u_t + \langle A\rangle u_x = \frac{1}{2}(\sigma_A^2 t_f)u_{xx},\\ &u(x,0) = \langle Q_0(x)^m\rangle. \end{aligned} \tag{21}$$

The solutions of (19) and (21), $u_1(x,t)$ and $u_2(x,t)$, respectively, are equal at $t = t_f$. Indeed, according to [14] we have

$$u_1(x,t_f) = \frac{1}{\sqrt{\pi}\xi_1(t_f)}\int_{-\infty}^{+\infty} \exp\left[-\left(\frac{x-\langle A\rangle t_f - \omega}{\xi_1(t_f)}\right)^2\right]\langle Q_0(\omega)^m\rangle \, d\omega, \tag{22}$$

where

$$\xi_1(t_f) = 2\left[\int_0^{t_f}(\sigma_A^2 s)\, ds\right]^{1/2} = \sqrt{2}\sigma_A t_f.$$

On the other hand, the solution to (21) is also given by (22) with

$$\xi_2(t_f) = 2\left[\int_0^{t_f}[(\sigma_A^2 t_f)/2]\, ds\right]^{1/2}$$

instead of $\xi_1(t_f)$. Since $\xi_1(t_f) = \xi_2(t_f)$ then $u_1(x,t_f) = u_2(x,t_f)$.

Therefore (20) is more than a consistency condition: it guarantees the convergence of the method to the solution at $t = t_f$.

For this particular example (normal velocity), we have shown that each moment of the solution to (1), $\langle Q(x,t)^m\rangle$, satisfies the advection–diffusion Eq. (17) with $\nu = \nu(t)$. As a consequence, the probability density function for the random solution $Q(x,t)$, $f_Q(q;x,t)$, also satisfies the advection–diffusion equation

$$\begin{aligned} &(f_Q)_t + \langle A\rangle (f_Q)_x = \nu(t)\ (f_Q)_{xx},\\ &f_Q(q;x,0) = f_{Q_0}(q;x). \end{aligned} \tag{23}$$

Indeed, the Fourier transform of $f_Q(q;x,t)$, under the assumption that the probability density function is uniquely determined by its moments (see e.g., [8] for conditions for uniqueness in the problems of moments), is

$$\widehat{f_Q}(\omega;x,t) = \sum_{j=0}^{\infty}\frac{(i\omega)^j}{j!}\langle Q^m(x,t)\rangle, \tag{24}$$

where $\langle Q^m(x,t)\rangle_t + \langle A\rangle\langle Q^m(x,t)\rangle_x = \nu(t)\ \langle Q^m(x,t)\rangle_{xx}$. Taking the derivative with respect to t and x in (24), we arrive at

$$\left(\widehat{f_Q}\right)_t + \langle A\rangle\left(\widehat{f_Q}\right)_x = \nu(t)\ \left(\widehat{f_Q}\right)_{xx}. \tag{25}$$

Since the variable ω does not appear in the derivatives, we can go back to the variable u and find (23). The respective initial condition follows from the probability density function of $Q_0(x)$.

4. The system of partial differential equations for the central moments

The central moments of a given random function $Q(x,t)$ are deterministic functions defined by $\mu_m = \langle (Q - \langle Q\rangle)^m\rangle$, $m \in \mathbb{N}$, $m \geqslant 2$. The most used central moment is the variance, $m = 2$, which was introduced by Gauss (1777–1855) as a measure of dispersion of the distribution of $Q(x,t)$. But high order central moments are also useful information concerning random variables [13,16]. In the following we show that the central moment $\mu_m(x,t)$, if sufficiently smooth, satisfies an advective–diffusive equation with the source term defined by the expectation and the central moments $\mu_{m-1}(x,t)$ and $\mu_{m-2}(x,t)$. Here, we extend the definition of central moments for $m \geqslant 0$ since $\mu_0 = 1$ and $\mu_1 = 0$.

We may use algebraic manipulations to show that

(i) If $k \leqslant m-2$ then

$$\binom{m}{k+2}(k+1)(k+2) = \binom{m}{k}(m-k)(m-k-1). \tag{26}$$

(ii) If $k \leqslant m-1$ then

$$\binom{m}{k+2}(k+1) = \binom{m}{k}(m-k). \tag{27}$$

(iii)

$$\mu_m = \langle Q^m\rangle - \sum_{k=2}^{m-1}\binom{m}{k}\mu_k\langle Q\rangle^{m-k} - \langle Q\rangle^m. \tag{28}$$

Proposition 5. *Let Z(x,t) be a random function whose statistical moments satisfy (17), i.e., the advective–diffusive equations:*

$$\langle Z^m\rangle_t + \langle A\rangle\langle Z^m\rangle_x = \nu\langle Z^m\rangle_{xx}, \tag{29}$$

$m\in\mathbb{N}$, $m\geqslant 1$. Then the central moments, $\mu_m(x,t) = \langle[Z-\langle Z\rangle]^m\rangle$, $m\in\mathbb{N}$, $m\geqslant 2$, satisfy the advective–diffusive equations with source term:

$$\mu_{m,t} + \langle A\rangle\mu_{m,x} - \nu\mu_{m,xx} = 2m\nu\mu_{m-1,x}\,\langle Z\rangle_x + m(m-1)\nu\mu_{m-2}\langle Z\rangle_x^2, \tag{30}$$

where $\mu_0 = 1$ and $\mu_1 = 0$.

Proof. The proof is based on the induction principle. Since $\mu_2(x,t) = \langle Z^2(x,t)\rangle - \langle Z(x,t)\rangle^2$, $\mu_1(x,t) = 0$ and $\mu_0(x,t) = 1$, direct substitution and derivations show (30) for $k = 2$. As the induction hypothesis we assume that (30) is true for $k = 3:(m-1)$, and our task is to prove that (30) is true for $k = m$. From (28) we have

$$\mu_m(x,t) = \langle Z^m\rangle - \sum_{k=2}^{m-1}\binom{m}{k}\mu_k\langle Z\rangle^{m-k} - \langle Z\rangle^m.$$

By differentiating this expression with respect to t and x, grouping conveniently the terms, and using (29) we arrive at

$$\begin{aligned}\mu_{m,t} + \langle A\rangle\mu_{m,x} - \nu\mu_{m,xx} = &-\sum_{k=2}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\left\{\mu_{k,t} + \langle A\rangle\mu_{k,x} - \nu\mu_{k,xx}\right\} + 2\;\nu\;\sum_{k=2}^{m-1}\binom{m}{k}\;(m-k)\;\mu_{k,x}\;\langle Z\rangle^{m-k-1}\;\langle Z\rangle_x\\ &+\nu\sum_{k=2}^{m-2}\binom{m}{k}\;(m-k)\;(m-k-1)\;\mu_k\;\langle Z\rangle^{m-k-2}\;(\langle Z\rangle_x)^2 + \nu\;m\;(m-1)\;\langle Z\rangle^{m-2}\langle Z\rangle_x^2.\end{aligned} \tag{31}$$

Using the induction hypothesis in the first sum in (31), and separating the last term of the second and third sums, we obtain

$$\begin{aligned}\mu_{m,t} + \langle A\rangle\mu_{m,x} - \nu\mu_{m,xx} = &-\sum_{k=2}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\left\{2k\nu\mu_{k-1,x}\langle Z\rangle_x + k(k-1)\nu\mu_{k-2}(\langle Z\rangle_x)^2\right\}\\ &+2\;\nu\;\sum_{k=2}^{m-2}\binom{m}{k}\;(m-k)\;\mu_{k,x}\;\langle Z\rangle^{m-k-1}\;\langle Z\rangle_x + \nu\sum_{k=2}^{m-3}\binom{m}{k}\;(m-k)\;(m-k-1)\;\mu_k\;\langle Z\rangle^{m-k-2}\;\langle Z\rangle_x^2\\ &+2\;m\;\nu\;\mu_{m-1,x}\;\langle Z\rangle_x + m\;(m-1)\;\nu\;\mu_{m-2}\;\langle Z\rangle_x^2 + \underbrace{\nu\;m\;(m-1)\;\langle Z\rangle^{m-2}\langle Z\rangle_x^2}_{\text{equal the first sum with } k=2},\end{aligned}$$

or, equivalently,

$$\begin{aligned}\mu_{m,t} + \langle A\rangle\mu_{m,x} - \nu\mu_{m,xx} = &\,2m\nu\mu_{m-1,x}\langle Z\rangle_x + m(m-1)\nu\mu_{m-2}\langle Z\rangle_x^2 - \nu\sum_{k=3}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\left\{2k\mu_{k-1,x}\langle Z\rangle_x + k(k-1)\mu_{k-2}\langle Z\rangle_x^2\right\}\\ &+\nu\;\sum_{k=2}^{m-2}\binom{m}{k}\;(m-k)\;2\;\mu_{k,x}\;\langle Z\rangle^{m-k-1}\;\langle Z\rangle_x + \nu\sum_{k=2}^{m-3}\binom{m}{k}\;(m-k)\;(m-k-1)\;\mu_k\;\langle Z\rangle^{m-k-2}\;\langle Z\rangle_x^2.\end{aligned} \tag{32}$$

To show that the three sums on the right side of (32) are zero, we open the first one of them

$$\begin{aligned}&\sum_{k=3}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\left\{2\;k\;\mu_{k-1,x}\langle Z\rangle_x + k(k-1)\mu_{k-2}\langle Z\rangle_x^2\right\}\underbrace{=}_{\mu_1=0}\\ &=\sum_{k=3}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\;2k\mu_{k-1,x}\langle Z\rangle_x + \sum_{k=4}^{m-1}\binom{m}{k}\langle Z\rangle^{m-k}\;k(k-1)\mu_{k-2}\langle Z\rangle_x^2\\ &=2\sum_{k=2}^{m-2}\binom{m}{k+1}(k+1)\langle Z\rangle^{m-k-1}\mu_{k,x}\langle Z\rangle_x + \sum_{k=2}^{m-3}\binom{m}{k+2}(k+1)(k+2)\langle Z\rangle^{m-k-2}\mu_k\langle Z\rangle_x^2\underbrace{=}_{\text{using (26) and (27)}}\\ &=2\sum_{k=2}^{m-2}\binom{m}{k}(m-k)\mu_{k,x}\langle Z\rangle^{m-k-1}\langle Z\rangle_x + \sum_{k=2}^{m-3}\binom{m}{k}(m-k)(m-k-1)\mu_k\langle Z\rangle^{m-k-2}\langle Z\rangle_x^2.\end{aligned}$$

Therefore, from (32) we arrive at (30). □

Remark 6. In Section 3 we have shown that the numerical method (16), for the moments, is stable and consistent with (17). Since we have used the same method (16) to compute the central moments, we conclude that the method for the central moments is stable and consistent with (30).

5. Computational tests

In this section, we present some examples to assess our approach. In Examples 1 and 2 the initial condition allows exact statistical moments of the solution. We use Riemann initial conditions defined by bivariate normal distributions; in this case the solutions for the moments are given by (4). In order to investigate the influence of the randomness we use two models: in Example 1 the velocity, A, is normally distributed, and in Example 2 the velocity is lognormally distributed. In both cases we compare the exact solutions, given by (4), with the solutions yielded by the numerical scheme (16) for some statistical moments. In Example 3 we apply our method in the problem (1) where the initial condition is a normal random function and the transport velocity is a normal random variable. The numerical experiments presented in this section were done in double precision with some MATLAB codes on a 3.0 GHz Pentium 4 with 512 Mb of memory.

Example 1. Let us consider the random Riemann problem (1)–(2) where the random velocity is normally distributed, $A \sim N(1.0, 0.8)$, and the random variables Q_L and Q_R have a bivariate normal distribution defined by: $\langle Q_L \rangle = 1.0$ (mean of Q_L); $\langle Q_R \rangle = 0.0$ (mean of Q_R); $\sigma_L = 0.4$ (standard deviation of Q_L); $\sigma_R = 0.5$ (standard deviation of Q_R); and $\rho = 0.4$ (correlation coefficient between Q_L and Q_R). In Fig. 2 we compare the exact values for the mean, variance, 3rd central moment, and 4th central moment with the computations using (16) at $t_f = 0.4$, and Δt and Δx satisfying (20).

Example 2. To check the influence of the velocity distribution we consider the random Riemann problem (1)–(2) in which the random velocity is lognormally distributed, $A = \exp(\xi)$, $\xi \sim N(0.5, 0.35)$. The initial condition (Q_L, Q_R) has a bivariate normal distribution defined by: $\langle Q_L \rangle = 1.0$; $\langle Q_R \rangle = 0.15$; $\sigma_L = 0.36$; $\sigma_R = 0.25$; and $\rho = 0.4$. Taking the lognormal distribution, $A = \exp(\xi)$, $\xi \sim N(\mu_\xi, \sigma_\xi)$, in (18) we obtain

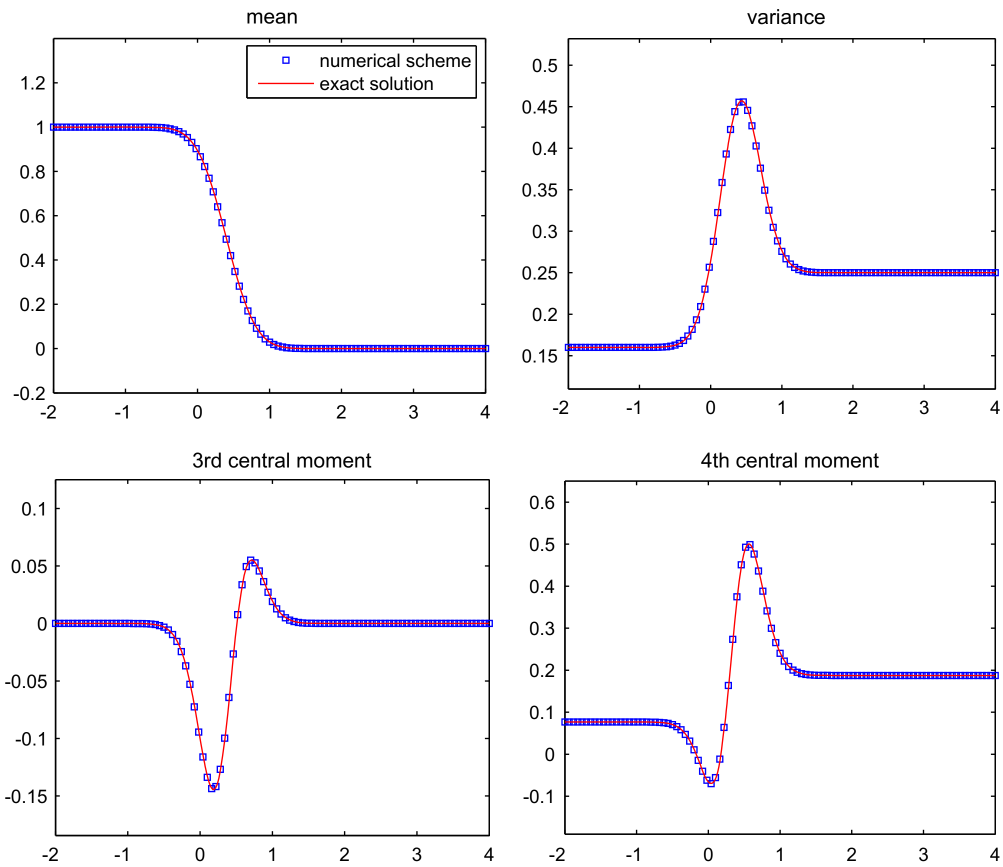

Fig. 2. $A \sim N(1.0, 0.8)$, $\Delta x = 0.01$, $\Delta t = 0.000195$, and $t_f = 0.4$.

$$v(x,t) = \frac{\sigma_\xi^2 \left(\frac{x}{t}\right)\left(\frac{x}{t} - \langle A \rangle t\right)}{\left(\sigma_\xi^2 - \mu_\xi\right) + \ln\left(\frac{x}{t}\right)}. \tag{33}$$

This mean that it is not possible to find constants Δx and Δt such that $(\Delta x^2)/(4\Delta t) = v$, the consistency condition. Moreover, the diffusion coefficient (33) may assume negative values loosing the physical meaning. Thus, although these arguments are not conclusive, they suggest that an advective–diffusive equation is not a good model to the moments of the solution to (1) with a lognormal velocity. If we use (20) as in the previous example the results loose quality as shown in Fig. 3.

Example 3. In this example we test our method for the random partial differential equation (1) in which A is normal, $A \sim N(-0.5, 0.6)$, and $Q_0(x)$ is a normal random function with mean

$$\langle Q_0(x) \rangle = \begin{cases} 1, & x \in (1.4,\ 2.2), \\ e^{-20(x-0.25)^2}, & \text{otherwise}, \end{cases} \tag{34}$$

and covariance $\mathrm{Cov}(x,\tilde{x}) = \sigma^2 \exp(-\beta|x-\tilde{x}|)$, where $\mathrm{Var}[Q_0(x)] = \sigma^2$ is constant and $\beta > 0$ governs the decay rate of the spatial correlation. We use $\beta = 0.3$ and $\sigma^2 = 0.16$. The numerical results are compared with the Monte Carlo method using suites of realizations of A and $Q_0(x)$, where A and $Q_0(x)$ are statistically independents. Observe that each realization $A(\omega)$ and $Q_0(x,\omega)$ yields analytical solution given by $Q(x,t,\omega) = Q_0(x - A(\omega)t, \omega)$. To generate the realizations required by Monte Carlo simulations we use random numbers generator of MATLAB. Comparisons with the Monte Carlo method, with 30,000 realizations, are plotted in Fig. 4.

6. Conclusions

In this paper, we have used the Godunov ideas to obtain a numerical scheme for the statistical moments of the solution of the one-dimensional random linear transport equation. We consider the velocity as a random variable and the initial condition as a random function. We have used an explicit solution of the random Riemann problem to evolve in the REA algo-

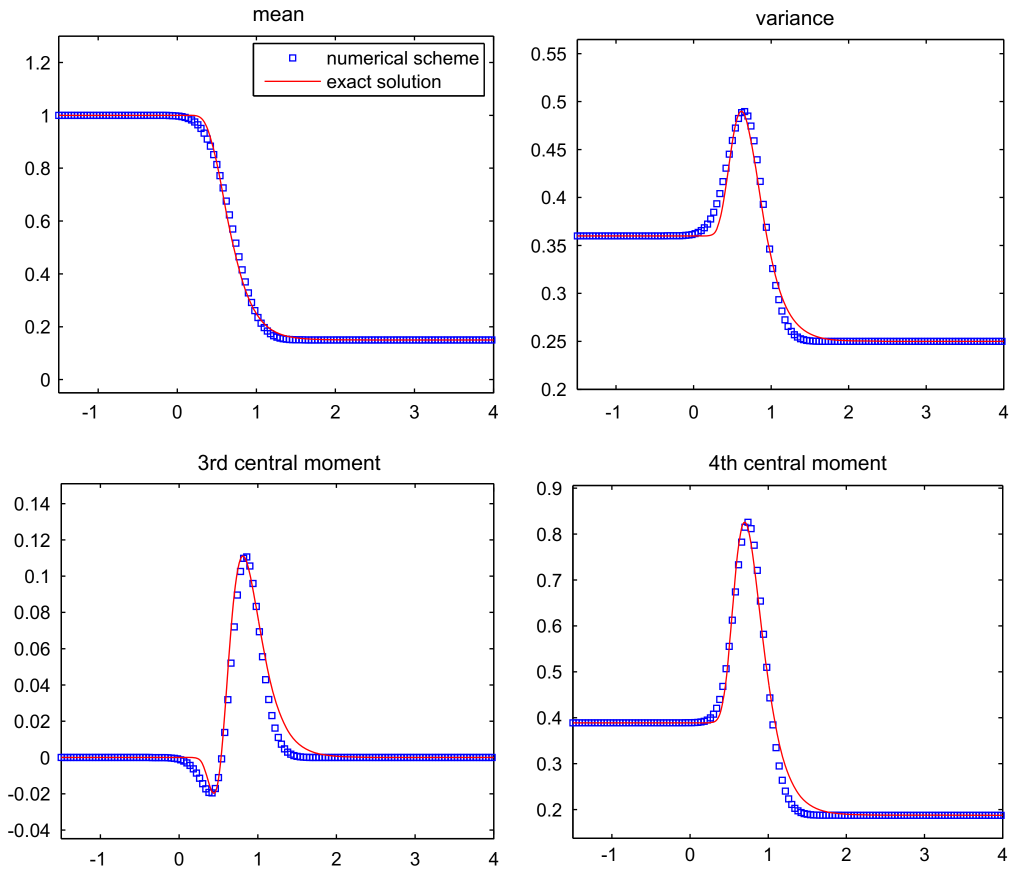

Fig. 3. $A = \exp(\xi)$, $\xi \sim N(0.5, 0.35)$, $\Delta x = 0.01$, $\Delta t = 0.000312$, and $t_f = 0.4$.

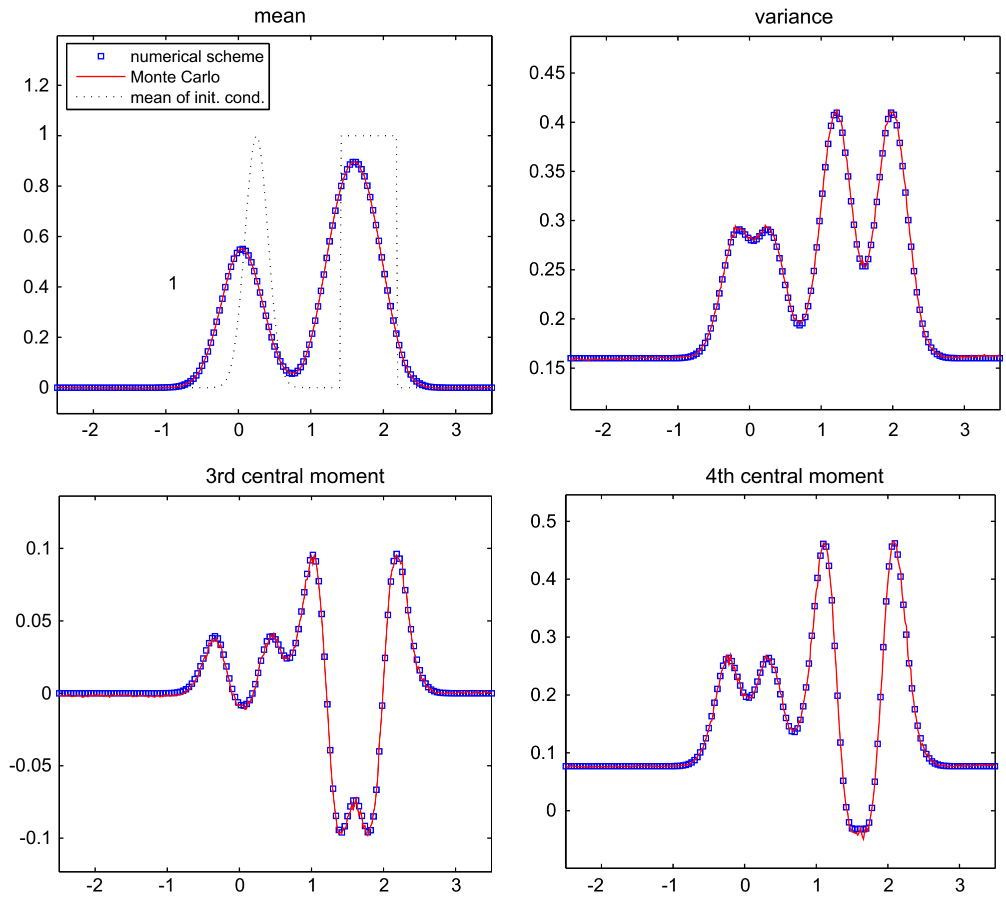

Fig. 4. $A \sim N(-0.5, 0.6)$, $\Delta x = 0.02$, $\Delta t = 0.000138$, and $t_f = 0.4$.

rithm. Moreover, we have shown that the scheme is stable and consistent with an advective–diffusive equation. A particular Riemann problem solution is used to find the diffusion coefficient of the differential equations for the statistical moments. Also, we have obtained the differential equations for the central moments of the solution. Computational tests have illustrated our theoretical results.

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